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BACKWARD STOCHASTIC NONLINEAR VOLTERRA INTEGRAL EQUATIONS WITH LOCAL LIPSCHITZ DRIFT

BY

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Abstract. In this paper, we study backward stochastic nonlinear Volterra integral equations. Under a local Lipschitz continuity condition on the drift, we prove the existence and uniqueness result. We also establish a stability property for this kind of equations.

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1. INTRODUCTION

A linear version of backward stochastic differential equations (BSDE's in short) was first considered by Bismut ([7], [8]) in the context of optimal stochastic control. Nonlinear BSDE's have been independently introduced by Pardoux and Peng [23] and Duffie and Epstein [10]. These equations were intensively investigated in the last years. The main reason for this great interest in these equations is because of their connections with many other fields of research such as: mathematical finance (see El Karoui et al. [13]), stochastic control and stochastic games (see Hamadène and Lepeltier [17]). These equations also provide probabilistic interpretation for solutions to both elliptic and parabolic nonlinear partial differential equations (see Pardoux and Peng [24], Peng [26]). Indeed, coupled with a forward SDE, such BSDE's give an extension of the celebrate Feynman–Kac formula to the nonlinear case.

The classical condition on the drift for proving the existence and uniqueness result is a global Lipschitz one. Many authors have attempted to relax this condition. For instance, several works treat BSDE's with continuous or local Lipschitz drift (see Hamadène [15], [16], Lepeltier and San Martin [20], N'Zi and Ouknine [22] and the references therein). In the one-dimensional case, the essential tool is the comparison-theorem technique. In the multidimensional case, the improvements of the Lipschitz condition on the generator concern, generally, the variable y only and the conditions considered are global. It seems that the first works treating multidimensional BSDE's with both local conditions on the drift and only square-integrable terminal data are Bahlali [2], [3]. This author considered BSDE's with locally Lipschitz coefficients both in y and z. This study has been continued by Bahlali et al. [4], Aman and N'Zi [1] and Essaky et al. [14].

Recently, backward stochastic nonlinear Volterra integral equations (BSNVIE's in short) have been studied by Lin [21] under the global Lipschitz condition on the drift. His work is a continuation of a previous one of Hu and Peng [18] where backward semilinear stochastic evolution equations with values in a complete separable Hilbert space have been considered. More precisely, Lin [21] gives an existence and uniqueness result for the following nonlinear BSDE of Volterra type:

$$(1.1)\quad Y(t)+\int_t^T f(t, s, Y(s), Z(t, s))\,ds+\int_t^T [g(t, s, Y(s))+Z(t, s)]\,dW(s) = \xi.$$

On the other hand, ordinary stochastic Volterra integral equations have been investigated by Berger and Mizel [5], [6], Pardoux and Protter [25], Protter [27], Kolodh [19] and have found applications in mathematical finance (see [9] and [11]).

In this paper, we are concerned with equation (1.1) and our aim is to weaken the global Lipschitz condition on the drift to a local one. The paper is organized as follows. In Section 2, we give essential notions on backward stochastic nonlinear Volterra equations and Section 3 deals with the main result. Finally, Section 4 is devoted to a stability result.

2. ASSUMPTIONS AND FORMULATION OF THE PROBLEM

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\leq t\leq T}, \boldsymbol{P})$ be a filtered probability space satisfying the usual conditions and $\{W(t), t\in[0, T]\}$ the d-dimensional standard Brownian motion defined on it.

Define $\mathcal{D} = \{(t, s)\in \boldsymbol{R}_+^2;\ 0\leq t\leq s\leq T\}$ and denote by $\mathcal{P}$ the $\bar{\sigma}$-algebra of $\mathcal{F}_{t\vee s}$-progressively measurable subsets of $\Omega\times\mathcal{D}$.

Let $M^2(t, T; \boldsymbol{R}^k)$ (resp. $M^2(\mathcal{D}; \boldsymbol{R}^{k\times d})$) be the set of $\boldsymbol{R}^k$-valued (resp. $\boldsymbol{R}^{k\times d}$-valued), $\mathcal{F}_{t\vee s}$-progressively measurable processes which are square-integrable with respect to $\boldsymbol{P}\otimes\lambda\otimes\lambda$ (here λ denotes Lebesgue measure over $[0, T]$). For $X\in\boldsymbol{R}^k$, $|X|$ will denote its Euclidean norm. An element $Y\in\boldsymbol{R}^{k\times d}$ will be considered as a $k\times d$-matrix; its Euclidean norm is given by $|Y| = \sqrt{\mathrm{Tr}(YY^*)}$ and $\langle Y, Z\rangle = \mathrm{Tr}(YZ^*)$.

$\mathcal{B}_k$ stands for the Borel σ-algebra of $\boldsymbol{R}^k$.

Moreover, we are given the following objects and assumptions:

(A1) $f\colon \Omega\times\mathcal{D}\times\boldsymbol{R}^k\times\boldsymbol{R}^{k\times d}\to\boldsymbol{R}^k$ is a $(\mathcal{P}\otimes\mathcal{B}_k\otimes\mathcal{B}_{k\times d}/\mathcal{B}_k)$-measurable function satisfying:

(i) $f(\cdot, \cdot, 0, 0) \in M^2(\mathscr{D}; \boldsymbol{R}^k)$;

(ii) there exist two constants $K > 0$ (K sufficiently large) and $0 \leqslant \alpha < 1$ such that

$$|f(t, s, y, z)| \leqslant K(1+|y|+|z|)^{\alpha}, \ \boldsymbol{P}\text{-a.s., a.e. } (t, s) \in \mathscr{D};$$

(iii) for every $N \in \boldsymbol{N}$, there exists a constant $L_N > 0$ such that

$$|f(t, s, y, z) - f(t, s, y', z')| \leqslant L_N |y-y'| + K|z-z'|$$

for all $|y| \leqslant N$, $|y'| \leqslant N$, for all $(t, s) \in \mathscr{D}$, $z \in \boldsymbol{R}^{k \times d}$, $z' \in \boldsymbol{R}^{k \times d}$, where K is the constant in (A1) (ii).

(A2) $g\colon \Omega \times \mathscr{D} \times \boldsymbol{R}^k \to \boldsymbol{R}^{k \times d}$ is a $(\mathscr{P} \otimes \mathscr{B}_k / \mathscr{B}_{k \times d})$-measurable function which satisfies:

(i) $g(\cdot, \cdot, 0) \in M^2(\mathscr{D}; \boldsymbol{R}^{k \times d})$;

(ii) $|g(t, s, y) - g(t, s, y')| \leqslant K|y-y'|$ for all $y, y' \in \boldsymbol{R}^k$ and for all $(t, s) \in \mathscr{D}$, where K is the constant in (A1) (ii).

(A3) ξ is a square-integrable k-dimensional $\mathscr{F}_T$-measurable random vector.

Remark 2.1. Note that (A1) and (A2) imply

$$f(\cdot, \cdot, Y(\cdot), Z(\cdot, \cdot)) \in M^2(\mathscr{D}; \boldsymbol{R}^k) \quad \text{and} \quad g(\cdot, \cdot, Y(\cdot)) \in M^2(\mathscr{D}; \boldsymbol{R}^{k \times d})$$

whenever $Y \in M^2(t, T; \boldsymbol{R}^k)$, $Z \in M^2(\mathscr{D}; \boldsymbol{R}^{k \times d})$.

DEFINITION 2.2. A *solution to BSDE of Volterra type* with data (ξ, f, g) is a pair of $\mathscr{F}_{t \vee s}$-adapted processes $\{(Y(s), Z(t, s)); (t, s) \in \mathscr{D}\}$ with values in $M^2(t, T; \boldsymbol{R}^k) \times M^2(\mathscr{D}; \boldsymbol{R}^{k \times d})$ which solves (1.1).

3. EXISTENCE AND UNIQUENESS

Before stating the main result, let us give some preliminaries.

LEMMA 3.1. *Let f denote a process satisfying assumption* (A1). *Then there exists a sequence of processes $(f_n)_{n \geqslant 1}$ such that, for every $n \geqslant 1$, f_n is $(\mathscr{P} \otimes \mathscr{B}_k \otimes \mathscr{B}_{k \times d} / \mathscr{B}_k)$-measurable, Lipschitzian, satisfies* (A1) (i), (A1) (ii) *and $\varrho_N(f_n - f) \to 0$ as $n \to +\infty$ for every fixed N, where*

$$\varrho_N(f) = \boldsymbol{E}\Big(\int_{\mathscr{D}} \sup_{|y| \leqslant N} \sup_{z \in \boldsymbol{R}^{k \times d}} |f(t, s, y, z)|^2 \, dt\, ds\Big)^{1/2}$$

Proof. Let ψ_n be a sequence of smooth functions with support in the ball $B(0, n+1)$ such that $\psi_n = 1$ in the ball $B(0, n)$ and $\sup \psi_n = 1$. One can easily show that the sequence $(f_n)_{n \geqslant 1}$ of truncated functions defined by $f_n = f\psi_n$ satisfies all the properties quoted above. ■

Let $(f_n)_{n \geqslant 1}$ be associated with f by Lemma 3.1. By the results of Lin [21], for every $n \geqslant 1$, there exists a unique couple of processes $\{(Y_n(s), Z_n(t, s))\colon (t, s) \in \mathscr{D}\}$, an element of $M^2(t, T; \boldsymbol{R}^k) \times M^2(\mathscr{D}; \boldsymbol{R}^{k \times d})$ solution to the BSDE of

Volterra type with data (ξ, f_n, g). We build the unique solution to equation (1.1) by studying convergence of the sequence $\{(Y_n(s), Z_n(t, s)): (t, s)\in\mathscr{D}\}$.

LEMMA 3.2. *Assume* (A1)–(A3) *hold true. Then there exists a constant* $C > 0$, *depending only on* T, K *and* ξ *such that for every* $n \geqslant 1$

$$E\int_t^T |Y_n(s)|^2\,ds + E\int_t^T ds\int_s^T |Z_n(s, u)|^2\,du \leqslant C \quad \text{for all } t\in[T-\eta, T],$$

where $\eta < 1/24K^2$.

Proof. Since $\{(Y_n(s), Z_n(t, s)): (t, s)\in\mathscr{D}\}$ is the unique solution to the BSDE of Volterra type with data (ξ, f_n, g), we have

$$(3.1)\qquad Y_n(t)+\int_t^T f_n\big(t, s, Y_n(s), Z_n(t, s)\big)\,ds + \int_t^T \big[g\big(t, s, Y_n(s)\big)+Z_n(t, s)\big]\,dW(s) = \xi.$$

Let $\mathscr{D}_\eta = \{(t, s)\colon T-\eta \leqslant t \leqslant s \leqslant T\}$, where η will be precised later. By Lemma 2.1 of [21], for every $(t, s)\in\mathscr{D}_\eta$, we have

$$\begin{aligned}(3.2)\qquad & E\,|Y_n(s)|^2 + E\int_s^T |Z_n(s, u)|^2\,du \\
&= E\,|\xi|^2 - 2E\int_s^T \big\langle f_n\big(s, u, Y_n(u), Z_n(s, u)\big), Y_n(u)\big\rangle\,du \\
&\quad - 2E\int_s^T \big\langle f_n\big(s, u, Y_n(u), Z_n(s, u)\big), A_n(s, u)\big\rangle\,du \\
&\quad - 2E\int_s^T \big\langle g\big(s, u, Y_n(u)\big), Z_n(s, u)\big\rangle\,du - E\int_s^T \big|g\big(s, u, Y_n(u)\big)\big|^2\,du \\
&\leqslant E\,|\xi|^2 + 2E\int_s^T \big|\big\langle f_n\big(s, u, Y_n(u), Z_n(s, u)\big), Y_n(u)\big\rangle\big|\,du \\
&\quad + 2E\int_s^T \big|\big\langle f_n\big(s, u, Y_n(u), Z_n(s, u)\big), A_n(s, u)\big\rangle\big|\,du \\
&\quad + 2E\int_s^T \big|\big\langle g\big(s, u, Y_n(u)\big), Z_n(s, u)\big\rangle\big|\,du,\end{aligned}$$

where

$$A_n(s, u) = \int_u^T \Big(f_n\big(u, v, Y_n(v), Z_n(u, v)\big) - f_n\big(s, v, Y_n(v), Z_n(s, v)\big)\Big)\,dv.$$

Using assumptions (A1), (A2) on f_n and g, and the Young inequality $2ab \leqslant \beta a^2 + b^2/\beta$ for every $\beta > 0$, we derive the following inequalities:

$$(3.3)\quad 2\left|\left\langle f_n(s, u, Y_n(u), Z_n(s, u)), Y_n(u)\right\rangle\right| \leqslant 2\left|f_n(s, u, Y_n(u), Z_n(s, u))\right| |Y_n(u)|$$
$$\leqslant \frac{1}{\beta_1}\left|f_n(s, u, Y_n(u), Z_n(s, u))\right|^2 + \beta_1 |Y_n(u)|^2$$
$$\leqslant \frac{3K^2}{\beta_1}(1+|Y_n(u)|^2+|Z_n(s, u)|^2)+\beta_1 |Y_n(u)|^2$$
$$\leqslant \left(\beta_1+\frac{3K^2}{\beta_1}\right)|Y_n(u)|^2+\frac{3K^2}{\beta_1}|Z_n(s, u)|^2+\frac{3K^2}{\beta_1}.$$

Since

$$2\left|\left\langle f_n(s, u, Y_n(u), Z_n(s, u)), A_n(s, u)\right\rangle\right|$$
$$\leqslant \frac{1}{\beta_2}\left|f_n(s, u, Y_n(u), Z_n(s, u))\right|^2+\beta_2 |A_n(s, u)|^2$$
$$\leqslant \frac{3K^2}{\beta_2}(1+|Y_n(u)|^2+|Z_n(s, u)|^2)+\beta_2 |A_n(s, u)|^2$$

and

$$\beta_2 |A_n(s, u)|^2 \leqslant 2\beta_2 (T-u)\int_u^T \left|f_n(u, v, Y_n(v), Z_n(u, v))\right|^2 dv$$
$$+2\beta_2 (T-u)\int_u^T \left|f_n(s, v, Y_n(v), Z_n(s, v))\right|^2 dv$$
$$\leqslant 6\beta_2 (T-u) K^2 \int_u^T (1+|Y_n(v)|^2+|Z_n(u, v)|^2)\, dv$$
$$+6\beta_2 (T-u) K^2 \int_u^T (1+|Y_n(v)|^2+|Z_n(s, v)|^2)\, dv,$$

we have

$$(3.4)\quad 2\left|\left\langle f_n(s, u, Y_n(u), Z_n(s, u)), A_n(s, u)\right\rangle\right|$$
$$\leqslant \frac{3K^2}{\beta_2}(1+|Y_n(u)|^2+|Z_n(s, u)|^2)$$
$$+12\beta_2 (T-u)^2 K^2+12\beta_2 (T-u) K^2 \int_u^T |Y_n(v)|^2 dv$$
$$+6\beta_2 (T-u) K^2 \int_u^T (|Z_n(u, v)|^2+|Z_n(s, v)|^2)\, dv.$$

Now,

$$2|\langle g(s,u,Y_n(u)),Z_n(s,u)\rangle| \leqslant 2\beta_3 K^2|Y_n(u)|^2+\frac{1}{\beta_3}|Z_n(s,u)|^2 + 2\beta_3|g(s,u,0)|^2. \tag{3.5}$$

Combining (3.2)–(3.5), we get

$$\begin{aligned} & E|Y_n(s)|^2+E\int_s^T|Z_n(s,u)|^2\,du \\ & \leqslant E|\xi|^2+\left(\beta_1+\frac{3K^2}{\beta_1}+\frac{3K^2}{\beta_2}+2\beta_3K^2\right)E\int_s^T|Y_n(u)|^2\,du \\ & +12\beta_2K^2E\int_s^T(T-u)\,du\int_u^T|Y_n(v)|^2\,dv \\ & +6\beta_2K^2E\int_s^T(T-u)\,du\int_u^T\left(|Z_n(u,v)|^2+|Z_n(s,v)|^2\right)dv \\ & +\left(\frac{3K^2}{\beta_1}+\frac{3K^2}{\beta_2}+\frac{1}{\beta_3}\right)E\int_s^T|Z_n(s,u)|^2\,du+12\beta_2K^2E\int_s^T(T-u)^2\,du \\ & +\left(\frac{3K^2}{\beta_1}+\frac{3K^2}{\beta_2}\right)(T-s)+2\beta_3E\int_s^T|g(s,u,0)|^2\,du. \end{aligned} \tag{3.6}$$

Moreover, it is not difficult to show that for every process $\{h(s)\colon s\in[0,T]\}$ we have

$$E\int_s^T(T-u)\,du\int_u^T|h(v)|^2\,dv \leqslant \frac{1}{2}(T-s)^2E\int_s^T|h(u)|^2\,du. \tag{3.7}$$

So, by integrating (3.6) from t to T, we have

$$\begin{aligned} & E\int_t^T|Y_n(s)|^2\,ds+\int_t^T dsE\int_s^T|Z_n(s,u)|^2\,du \\ & \leqslant TE|\xi|^2+\left(\beta_1+\frac{3K^2}{\beta_1}+\frac{3K^2}{\beta_2}+2\beta_3K^2+6\beta_2K^2\eta^2\right)\int_t^T dsE\int_s^T|Y_n(u)|^2\,du \\ & +\left(\frac{3K^2}{\beta_1}+\frac{3K^2}{\beta_2}+\frac{1}{\beta_3}+3\beta_2K^2\eta^2\right)\int_t^T dsE\int_s^T|Z_n(s,u)|^2\,du \\ & +6\beta_2K^2T\int_t^T ds\int_s^T duE\int_u^T|Z_n(u,v)|^2\,dv+\beta_2K^2T^4+\left(\frac{3K^2}{\beta_1}+\frac{3K^2}{\beta_2}\right)T^2 \\ & +2\beta_3E\int_t^T ds\int_s^T|g(s,u,0)|^2\,du. \end{aligned}$$

Let us put

$$U_n(t) = \boldsymbol{E}\int_t^T |Y_n(s)|^2\,ds \quad \text{and} \quad V_n(t) = \boldsymbol{E}\int_t^T |Z_n(t, s)|^2\,ds.$$

By choosing $\beta_1 = \beta_2 = 24K^2$, $\beta_3 = 8$ and $\eta < 1/24K^2$, we deduce that there exist K_1 and K_2 depending only on ξ, T and K such that

$$U_n(t)+\frac{1}{2}\int_t^T V_n(s)\,ds \leqslant K_1\int_t^T U_n(s)\,ds+K_2\Big(1+\int_t^T ds\int_s^T V_n(u)\,du\Big). \tag{3.8}$$

From now on let $C = C(K, T, \xi)$ be a constant depending only on K, T, ξ which may vary from line to line. By virtue of (3.8), we have

$$-\frac{d}{dt}\big(\exp(K_1 t)\,\tilde{U}_n(t)\big)+\frac{1}{2}\exp(K_1 t)\,\tilde{V}_n(t) \leqslant C\Big(1+\int_t^T \exp(K_1 s)\,\tilde{V}_n(s)\,ds\Big), \tag{3.9}$$

where

$$\tilde{U}_n(t) = \int_t^T U_n(s)\,ds \quad \text{and} \quad \tilde{V}_n(t) = \int_t^T V_n(s)\,ds.$$

Integrating (3.9) from t to T, we obtain

$$\tilde{U}_n(t)\exp(K_1 t)+\frac{1}{2}\int_t^T \exp(K_1 s)\,\tilde{V}_n(s)\,ds \leqslant C\Big(1+\int_t^T ds\int_s^T \exp(K_1 r)\,\tilde{V}_n(r)\,dr\Big).$$

Consequently, by the Gronwall inequality, we infer that for every $n \geqslant 1$, $t \in [T-\eta, T]$

$$\int_t^T \tilde{V}_n(s)\,ds \leqslant C \quad \text{and} \quad \tilde{U}_n(t) \leqslant C. \tag{3.10}$$

Putting (3.10) in (3.8), we obtain again from the Gronwall inequality that there exists a constant $C = C(\xi, T, K)$ such that for every $n \geqslant 1$, $t \in [T-\eta, T]$

$$\boldsymbol{E}\int_t^T |Y_n(s)|^2\,ds+\boldsymbol{E}\int_t^T ds\int_s^T |Z_n(s, u)|^2\,du \leqslant C. \ \blacksquare$$

THEOREM 3.3. *Assume* (A1)–(A3) *hold true. If*

$$\lim_{N\to+\infty}\frac{1}{(2L_N+2L_N^2)\,N^{2(1-\alpha)}}\exp\big[(2L_N+2L_N^2)\,T\big] = 0, \tag{A}$$

then there is a unique process $\{(Y(s), Z(t, s)): (t, s) \in \mathscr{D}\}$ *with values in* $M^2(t, T; \boldsymbol{R}^k)\times M^2(\mathscr{D}; \boldsymbol{R}^{k\times d})$ *solution of equation* (1.1).

Before proving Theorem 3.3, let us make the following

Remark 3.4. The condition (A) is fulfilled if there exists $L \geqslant 0$ such that

$$(2L_N+2L_N^2)\,T \leqslant L+(1-\alpha)\log N.$$

Proof of Theorem 3.3.

Uniqueness. Let $\{(Y(s), Z(t,s)) : (t,s)\in\mathscr{D}\}$ and $\{(Y'(s), Z'(t,s)) : (t,s)\in\mathscr{D}\}$ be two solutions of equation (1.1). Define

$$\Delta Y(s) = Y(s) - Y'(s), \qquad \Delta Z(t,s) = Z(t,s) - Z'(t,s),$$
$$\Delta f(t,s) = f(t,s,Y(s),Z(t,s)) - f(t,s,Y'(s),Z'(t,s)),$$
$$\Delta g(t,s) = g(t,s,Y(s)) - g(t,s,Y'(s)).$$

For every $N \geqslant 1$, we set

$$A^N = \{(\omega, s, u)\in\Omega\times\mathscr{D}_\eta,\ |Y(s)|+|Z(s,u)|+|Y'(u)|+|Z'(s,u)| \geqslant N\},$$
$$\bar{A}^N = (\Omega\times\mathscr{D}_\eta)\backslash A^N.$$

In the sequel C is a positive constant depending only on K, T, and ξ which may vary from line to line.

We have

$$\Delta Y(s) + \int_s^T \Delta f(s,u)\,du + \int_s^T [\Delta g(s,u) + \Delta Z(s,u)]\,dW_u = 0.$$

Therefore, Lemma 2.1 in [21] yields

$$\begin{aligned}
(3.11)\qquad & E|\Delta Y(s)|^2 + E\int_s^T |\Delta Z(s,u)|^2\,du \\
&= -2E\int_s^T \langle \Delta f(s,u), \Delta Y(u)\rangle\,du - 2E\int_s^T \langle \Delta f(s,u), A(s,u)\rangle\,du \\
&\quad - 2E\int_s^T \langle \Delta g(s,u), \Delta Z(s,u)\rangle\,du - E\int_s^T |\Delta g(s,u)|^2\,du \\
&\leqslant 2E\int_s^T |\Delta f(s,u)|\,|\Delta Y(u)|\,(\mathbf{1}_{A^N}(s,u) + \mathbf{1}_{\bar{A}^N}(s,u))\,du \\
&\quad + 2E\int_s^T |\Delta f(s,u)|\,|A(s,u)|\,(\mathbf{1}_{A^N}(s,u) + \mathbf{1}_{\bar{A}^N}(s,u))\,du \\
&\quad + 2E\int_s^T |\Delta g(s,u)|\,|\Delta Z(s,u)|\,du \\
&= J_1 + J_2 + J_3 + J_4 + J_5,
\end{aligned}$$

where

$$A(s,u) = \int_u^T (\Delta f(u,v) - \Delta f(s,v))\,dv.$$

In view of the assumptions (A1)–(A3), the Hölder inequality and the Young inequality, we derive the following inequalities:

$$\begin{aligned} J_1 &= 2E\int_s^T |\Delta f(s,u)|\,|\Delta Y(u)|\,\mathbf{1}_{A^N}(s,u)\,du \\ &\leqslant E\int_s^T |\Delta Y(u)|^2\,du + E\int_s^T |\Delta f(s,u)|^2\,\mathbf{1}_{A^N}(s,u)\,du \\ &\leqslant E\int_s^T |\Delta Y(u)|^2\,du \\ &\quad + 4K^2 E\int_s^T \big(1+|Y(u)|+|Z(s,u)|+|Y'(u)|+|Z'(s,u)|\big)^{2\alpha}\,\mathbf{1}_{A^N}(s,u)\,du. \end{aligned}$$

By virtue of the Hölder inequality and the Chebyshev inequality, we deduce that

$$J_1 \leqslant E\int_s^T |\Delta Y(u)|^2\,du + \frac{C}{N^{2(1-\alpha)}}, \tag{3.12}$$

$$\begin{aligned} J_2 &= 2E\int_s^T |\Delta f(s,u)|\,|\Delta Y(u)|\,\mathbf{1}_{\overline{A^N}}(s,u)\,du \\ &\leqslant 2E\int_s^T \big(L_N|\Delta Y(u)| + K|\Delta Z(s,u)|\big)\,|\Delta Y(u)|\,\mathbf{1}_{\overline{A^N}}(s,u)\,du \\ &\leqslant (2L_N+\beta_1)\int_s^T |\Delta Y(u)|^2\,du + \frac{K^2}{\beta_1} E\int_s^T |\Delta Z(s,u)|^2\,du, \end{aligned} \tag{3.13}$$

$$\begin{aligned} J_3 &= 2E\int_s^T |\Delta f(s,u)|\,|A(s,u)|\,\mathbf{1}_{A^N}(s,u)\,du \\ &\leqslant E\int_s^T |\Delta f(s,u)|^2\,\mathbf{1}_{A^N}(s,u)\,du + E\int_s^T |A(s,u)|^2\,du \\ &= I_1+I_2. \end{aligned}$$

We have

$$\begin{aligned} I_1 &\leqslant 4K^2 E\int_s^T \big(1+|Y(u)|+|Z(s,u)|+|Y'(u)|+|Z'(s,u)|\big)^{2\alpha}\,\mathbf{1}_{A^N}(s,u)\,du \\ &\leqslant \frac{C}{N^{2(1-\alpha)}}, \end{aligned}$$

$$\begin{aligned} I_2 &= E\int_s^T |A(s,u)|^2\,du \\ &\leqslant 2E\int_s^T (T-u)\,du\int_u^T |\Delta f(u,v)|^2\,dv + 2E\int_s^T (T-u)\,du\int_u^T |\Delta f(s,v)|^2\,dv. \end{aligned}$$

Let $\eta < 1/24K^2$. By (3.7), for $(s, u) \in \mathscr{D}_\eta$, we have

$$(3.14)\qquad I_2 \leqslant 2E\int_s^T (T-u)\,du \int_u^T |\Delta f(u, v)|^2 \left(\mathbf{1}_{A^N}(u, v) + \mathbf{1}_{\bar{A}^N}(u, v)\right) dv$$

$$+\eta^2 E\int_s^T |\Delta f(s, u)|^2 \left(\mathbf{1}_{A^N}(s, u) + \mathbf{1}_{\bar{A}^N}(s, u)\right) du$$

$$\leqslant 8K^2 E\int_s^T (T-u)\,du \int_u^T (1+|Y(v)|+|Z(u, v)|+|Y'(v)|$$

$$+|Z'(u, v)|)^{2\alpha} \mathbf{1}_{A^N}(u, v)\,dv + 4K^2\, TE\int_s^T du \int_u^T |\Delta Z(u, v)|^2\,dv$$

$$+4\eta^2 K^2 E\int_s^T \left(1+|Y(u)|+|Z(s, u)|+|Y'(u)|+|Z'(s, u)|\right)^{2\alpha} \mathbf{1}_{A^N}(s, u)\,du$$

$$+4\eta^2 L_N^2 E\int_s^T |\Delta Y(u)|^2\,du + 4\eta^2 K^2 E\int_s^T |\Delta Z(s, u)|^2\,du.$$

Using the Hölder inequality and the Chebyshev inequality, we deduce that

$$(3.15)\qquad J_3 \leqslant 4L_N^2 \eta^2 E\int_s^T |\Delta Y(u)|^2\,du + 4K^2 \eta^2 E\int_s^T |\Delta Z(s, u)|^2\,du$$

$$+\frac{(1+\eta^2)\,C}{N^{2(1-\alpha)}} + 4K^2\, TE\int_s^T du \int_u^T |\Delta Z(u, v)|^2\,dv,$$

$$J_4 = 2E\int_s^T |\Delta f(s, u)|\,|A(s, u)|\,\mathbf{1}_{\bar{A}^N}(s, u)\,du$$

$$\leqslant 2E\int_s^T \left(L_N|\Delta Y(u)| + K|\Delta Z(s, u)|\right)|A(s, u)|\,\mathbf{1}_{\bar{A}^N}(s, u)\,du$$

$$\leqslant L_N^2 E\int_s^T |\Delta Y(u)|^2\,du + (\beta_2+1)E\int_s^T |A(s, u)|^2\,du + \frac{K^2}{\beta_2} E\int_s^T |\Delta Z(s, u)|^2\,du.$$

Therefore, by virtue of (3.14), we have

$$(3.16)\qquad J_4 \leqslant [4(\beta_2+1)\eta^2+1]\,L_N^2 E\int_s^T |\Delta Y(u)|^2\,du$$

$$+\left[4(\beta_2+1)\eta^2 + \frac{1}{\beta_2}\right] K^2 E\int_s^T |\Delta Z(s, u)|^2\,du$$

$$+4(\beta_2+1)K^2\, TE\int_s^T \left(\int_u^T |\Delta Z(u, v)|^2\,dv\right) du + (\beta_2+1)\eta^2 \frac{C}{N^{2(1-\alpha)}}.$$

Now,

$$(3.17)\qquad J_5 = 2E\int_s^T |\Delta g(s,u)|\,|\Delta Z(s,u)|\,du$$

$$\leqslant \beta_3 E\int_s^T |\Delta g(s,u)|^2\,du + \frac{1}{\beta_3}E\int_s^T |\Delta Z(s,u)|^2\,du$$

$$\leqslant \beta_3 E\int_s^T |\Delta Y(u)|^2\,du + \frac{K^2}{\beta_3}E\int_s^T |\Delta Z(s,u)|^2\,du.$$

By combining (3.11)–(3.17) and integrating from t to T, we obtain

$$E\int_t^T |\Delta Y(s)|^2\,ds + \int_t^T ds E\int_s^T |\Delta Z(s,u)|^2\,du$$

$$\leqslant \big(1+2L_N+\beta_1+[4(\beta_2+2)\eta^2+1]\,L_N^2+\beta_3\big)E\int_t^T ds\int_s^T |\Delta Y(u)|^2\,du$$

$$+\left(\frac{1}{\beta_1}+\frac{1}{\beta_2}+\frac{1}{\beta_3}+4(\beta_2+2)\eta^2\right)K^2 E\int_t^T ds\int_s^T |\Delta Z(s,u)|^2\,du$$

$$+(1+\eta^2)(1+\beta_2)\frac{C}{N^{2(1-\alpha)}}+\big(4(\beta_2+2)K^2\,TE\int_t^T ds\int_s^T du\int_u^T |\Delta Z(u,v)|^2\big)\,dv.$$

Let us choose $\beta_1 = \beta_2 = \beta_3 = 8K^2$ and put

$$U(t) = E\int_t^T |\Delta Y(s)|^2\,ds \quad\text{and}\quad V(t) = E\int_t^T |\Delta Z(t,s)|^2\,ds.$$

Then we have

$$(3.18)\qquad U(t)+\frac{1}{2}\int_t^T V(s)\,ds \leqslant K_1\int_t^T U(s)\,ds+\frac{C}{N^{2(1-\alpha)}}+K_2\int_t^T ds\int_s^T V(u)\,du,$$

where $K_1 = 1+16K^2+2L_N+2L_N^2$, $K_2 = 4(8K^2+2)K^2\,T$.

It follows that

$$(3.19)\qquad -\frac{d}{dt}\big(\exp(K_1 t)\,\tilde U(t)\big)+\frac{1}{2}\exp(K_1 t)\,\tilde V(t)$$

$$\leqslant K_2\int_t^T \exp(K_1 s)\,\tilde V(s)\,ds+\frac{C}{N^{2(1-\alpha)}}\exp(K_1 t),$$

where

$$\tilde U(t) = \int_t^T U(s)\,ds \quad\text{and}\quad \tilde V(t) = \int_t^T V(s)\,ds.$$

Integrating (3.19) from t to T, we get

$$\exp(K_1 t)\tilde{U}(t)+\frac{1}{2}\int_t^T \exp(K_1 s)\tilde{V}(s)\,ds \tag{3.20}$$

$$\leqslant K_2\int_t^T ds\int_s^T \exp(K_1 u)\tilde{V}(u)\,du+\frac{C\exp(K_1 T)}{K_1 N^{2(1-\alpha)}}.$$

Therefore, the Gronwall inequality implies that for $t\in[T-\eta, T]$

$$\int_t^T \exp(K_1 s)\tilde{V}(s)\,ds \leqslant \frac{C}{(2L_N+2L_N^2)N^{2(1-\alpha)}}\exp[(2L_N+2L_N^2)T].$$

Passing to the limit on N, we deduce that for each $t\in[T-\eta, T]$ we have $\tilde{V}(t)=0$ and $\tilde{U}(t)=0$. Therefore, $Y(s)=Y'(s)$ and $Z(t, s)=Z'(t, s)$ for a.e. $(t, s)\in[T-\eta, T]\times[t, T]$.

For $t\in[T-2\eta, T-\eta]$, we have

$$\Delta Y(s)+\int_s^{T-\eta}\Delta f(s, u)\,du+\int_s^{T-\eta}[\Delta g(s, u)+\Delta Z(s, u)]\,dW_u=0.$$

Using the above procedure, we can deduce that for a.e. $(t, s)\in[T-2\eta, T-\eta]\times[t, T]$, $Y(s)=Y'(s)$ and $Z(t, s)=Z'(t, s)$ a.s. Hence, we can prove the uniqueness of (1.1).

Existence. For every $n, m\in N^*$ and $(t, s)\in\mathscr{D}_\eta$, let us set

$$A_{m,n}^N=\{(\omega, s, u)\in\Omega\times\mathscr{D}_\eta, |Y_n(u)|+|Z_n(s, u)|+|Y_m(u)|+|Z_m(s, u)|\geqslant N\},$$

$$\bar{A}_{m,n}^N=(\Omega\times\mathscr{D}_\eta)\backslash A_{m,n}^N.$$

Let

$$B_{m,n}(s, u)=\int_u^T\big(f_n(u, v, Y_n(v), Z_n(u, v))-f_m(u, v, Y_m(v), Z_m(u, v))\big)\,dv$$

$$-\int_u^T\big(f_n(s, v, Y_n(v), Z_n(s, v))-f_m(s, v, Y_m(v), Z_m(s, v))\big)\,dv.$$

We have

$$E|Y_n(s)-Y_m(s)|^2+E\int_s^T|Z_n(s, u)-Z_m(s, u)|^2\,du \tag{3.21}$$

$$=2E\int_s^T\langle f_n(s, u, Y_n(u), Z_n(s, u))-f_m(s, u, Y_m(u), Z_m(s, u)), Y_n(u)-Y_m(u)\rangle\,du$$

$$-2E\int_s^T\langle f_n(s, u, Y_n(u), Z_n(s, u))-f_m(s, u, Y_m(u), Z_m(s, u)), B_{m,n}(s, u)\rangle\,du$$

$$-2E\int_s^T \langle g(s,u,Y_n(u))-g(s,u,Y_m(u)),\ Z_n(s,u)-Z_m(s,u)\rangle\,du$$

$$-E\int_s^T |g(s,u,Y_n(u))-g(s,u,Y_m(u))|^2\,du$$

$$\leqslant 2E\int_s^T |f_n(s,u,Y_n(u),Z_n(s,u))-f_m(s,u,Y_m(u),Z_m(s,u))|\,|Y_n(u)-Y_m(u)|$$

$$\times\big(\mathbf{1}_{A^N_{m,n}}(s,u)+\mathbf{1}_{\bar{A}^N_{m,n}}(s,u)\big)\,du$$

$$+2E\int_s^T |f_n(s,u,Y_n(u),Z_n(s,u))-f_m(s,u,Y_m(u),Z_m(s,u))|\,|B_{m,n}(s,u)|$$

$$\times\big(\mathbf{1}_{A^N_{m,n}}(s,u)+\mathbf{1}_{\bar{A}^N_{m,n}}(s,u)\big)\,du$$

$$+2E\int_s^T |g(s,u,Y_n(u))-g(s,u,Y_m(u))|\,|Z_n(s,u)-Z_m(s,u)|\,du$$

$$=J_1+J_2+J_3+J_4+J_5.$$

By Lemma 3.2 and using the same calculations as in its proof, we have

$$J_1=2E\int_s^T |f_n(s,u,Y_n(u),Z_n(s,u))-f_m(s,u,Y_m(u),Z_m(s,u))| \tag{3.22}$$

$$\times|Y_n(u)-Y_m(u)|\,\mathbf{1}_{A^N_{m,n}}(s,u)\,du$$

$$\leqslant E\int_s^T |Y_n(u)-Y_m(u)|^2\,du+\frac{C}{N^{2(1-\alpha)}},$$

$$J_2=2E\int_s^T |f_n(s,u,Y_n(u),Z_n(s,u))-f_m(s,u,Y_m(u),Z_m(s,u))|$$

$$\times|Y_n(u)-Y_m(u)|\,\mathbf{1}_{\bar{A}^N_{m,n}}(s,u)\,du$$

$$\leqslant 2E\int_s^T |f_n(s,u,Y_n(u),Z_n(s,u))-f(s,u,Y_n(u),Z_n(s,u))|$$

$$\times|Y_n(u)-Y_m(u)|\,\mathbf{1}_{\bar{A}^N_{m,n}}(s,u)\,du$$

$$+2E\int_s^T |f(s,u,Y_n(u),Z_n(s,u))-f(s,u,Y_m(u),Z_m(s,u))|$$

$$\times|Y_n(u)-Y_m(u)|\,\mathbf{1}_{\bar{A}^N_{m,n}}(s,u)\,du$$

$$+2E\int_s^T \left|f(s,u,Y_m(u),Z_m(s,u))-f_m(s,u,Y_m(u),Z_m(s,u))\right|$$

$$\times |Y_n(u)-Y_m(u)|\,\mathbf{1}_{\overline{A}^N_{m,n}}(s,u)\,du$$

$$=I_1+I_2+I_3.$$

We have

$$I_1 \leqslant E\int_s^T |Y_n(u)-Y_m(u)|^2\,du+E\int_s^T \left|(f_n-f)(s,u,Y_n(u),Z_n(s,u))\right|^2 \mathbf{1}_{\overline{A}^N_{m,n}}(s,u)\,du,$$

$$I_2 \leqslant (2L_N+\beta_1)E\int_s^T |Y_n(u)-Y_m(u)|^2\,du+\frac{K^2}{\beta_1}E\int_s^T |Z_n(s,u)-Z_m(s,u)|^2\,du,$$

$$I_3 \leqslant E\int_s^T |Y_n(u)-Y_m(u)|^2\,du+E\int_s^T \left|(f_m-f)(s,u,Y_m(u),Z_m(s,u))\right|^2 \mathbf{1}_{\overline{A}^N_{m,n}}(s,u)\,du.$$

Hence

$$J_2 \leqslant (2L_N+\beta_1+2)E\int_s^T |Y_n(u)-Y_m(u)|^2\,du \tag{3.23}$$

$$+E\int_s^T \left|(f_n-f)(s,u,Y_n(u),Z_n(s,u))\right|^2 \mathbf{1}_{\overline{A}^N_{m,n}}(s,u)\,du$$

$$+E\int_s^T \left|(f_m-f)(s,u,Y_m(u),Z_m(s,u))\right|^2 \mathbf{1}_{\overline{A}^N_{m,n}}(s,u)\,du$$

$$+\frac{K^2}{\beta_1}E\int_s^T |Z_n(s,u)-Z_m(s,u)|^2\,du,$$

$$J_3 = 2E\int_s^T \left|f_n(s,u,Y_n(u),Z_n(s,u))-f_m(s,u,Y_m(u),Z_m(s,u))\right| \tag{3.24}$$

$$\times |B_{m,n}(s,u)|\,\mathbf{1}_{A^N_{m,n}}(s,u)\,du$$

$$\leqslant E\int_s^T \left|f_n(s,u,Y_n(u),Z_n(s,u))-f_m(s,u,Y_m(u),Z_m(s,u))\right|^2 \mathbf{1}_{A^N_{m,n}}(s,u)\,du$$

$$+E\int_s^T |B_{m,n}(s,u)|^2\,du$$

$$=I_4+I_5.$$

Using the Hölder inequality, the Chebyshev inequality and Lemma 3.2, we have

$$I_4 \leqslant \frac{C}{N^{2(1-\alpha)}}. \tag{3.25}$$

On the other hand,

$$I_5 \leqslant 2E\int_s^T (T-u)\,du \int_u^T \left|f_n\big(s, v, Y_n(v), Z_n(s, v)\big)-f_m\big(s, v, Y_m(v), Z_m(s, v)\big)\right|^2 dv$$

$$+\,2E\int_s^T (T-u)\,du \int_u^T \left|f_n\big(u, v, Y_n(v), Z_n(u, v)\big)-f_m\big(u, v, Y_m(v), Z_m(u, v)\big)\right|^2 dv.$$

By (3.7) we obtain

$$I_5 \leqslant \eta^2 E\int_s^T \left|f_n\big(s, u, Y_n(u), Z_n(s, u)\big)-f_m\big(s, u, Y_m(u), Z_m(s, u)\big)\right|^2$$

$$\times\big(\mathbf{1}_{A^N_{m,n}}(s, u)+\mathbf{1}_{\overline{A}^N_{m,n}}(s, u)\big)\,du+2E\int_s^T (T-u)\,du$$

$$\times\int_u^T \left|f_n\big(u, v, Y_n(v), Z_n(u, v)\big)-f_m\big(u, v, Y_m(v), Z_m(u, v)\big)\right|^2$$

$$\times\big(\mathbf{1}_{A^N_{m,n}}(u, v)+\mathbf{1}_{\overline{A}^N_{m,n}}(u, v)\big)\,dv.$$

Therefore, the Hölder inequality, the Chebyshev inequality and Lemma 3.2 yield

$$I_5 \leqslant \eta^2\frac{C}{N^{2(1-\alpha)}}+3\eta^2 E\int_s^T \left|(f_n-f)\big(s, u, Y_n(u), Z_n(s, u)\big)\right|^2 \mathbf{1}_{\overline{A}^N_{m,n}}(s, u)\,du \tag{3.26}$$

$$+\,3\eta^2 E\int_s^T \left|(f_m-f)\big(s, u, Y_m(u), Z_m(s, u)\big)\right|^2 \mathbf{1}_{\overline{A}^N_{m,n}}(s, u)\,du$$

$$+\,6E\int_s^T (T-u)\,du\int_u^T \left|(f_n-f)\big(u, v, Y_n(v), Z_n(u, v)\big)\right|^2 \mathbf{1}_{\overline{A}^N_{m,n}}(u, v)\,dv$$

$$+\,6E\int_s^T (T-u)\,du\int_u^T \left|(f_m-f)\big(u, v, Y_m(v), Z_m(u, v)\big)\right|^2 \mathbf{1}_{\overline{A}^N_{m,n}}(u, v)\,dv$$

$$+\,12\eta^2 L_N^2 E\int_s^T |Y_n(u)-Y_m(u)|^2\,du+6\eta^2 K^2 E\int_s^T |Z_n(s, u)-Z_m(s, u)|^2\,du$$

$$+\,12K^2 TE\int_s^T du\int_u^T |Z_n(u, v)-Z_m(u, v)|^2\,dv,$$

$$J_4 = 2E\int_s^T \big|f_n\big(s, u, Y_n(u), Z_n(s, u)\big)$$

$$-f_m\big(s, u, Y_m(u), Z_m(s, u)\big)\big|\,|B_{m,n}(s, u)|\,\mathbf{1}_{\overline{A}^N_{m,n}}(s, u)\,du$$

$$\leqslant 2E\int_s^T \left|(f_n-f)\big(s, u, Y_n(u), Z_n(s, u)\big)\right|\,|B_{m,n}(s, u)|\,\mathbf{1}_{\overline{A}^N_{m,n}}(s, u)\,du$$

$$+2E\int_s^T \left|f(s,u,Y_n(u),Z(s,u))-f(s,u,Y_m(u),Z_m(s,u))\right|$$

$$\times|B_{m,n}(s,u)|\,\mathbf{1}_{\overline{A}^N_{m,n}}(s,u)\,du$$

$$+2E\int_s^T \left|(f_m-f)(s,u,Y_m(u),Z_m(s,u))\right||B_{m,n}(s,u)|\,\mathbf{1}_{\overline{A}^N_{m,n}}(s,u)\,du$$

$$\leqslant E\int_s^T \left|(f_n-f)(s,u,Y_n(u),Z_n(s,u))\right|^2\mathbf{1}_{\overline{A}^N_{m,n}}(s,u)\,du$$

$$+E\int_s^T \left|(f_m-f)(s,u,Y_m(u),Z_m(s,u))\right|^2\mathbf{1}_{\overline{A}^N_{m,n}}(s,u)\,du$$

$$+L_N^2 E\int_s^T |Y_n(u)-Y_m(u)|^2\,du+(\beta_2+3)E\int_s^T |B_{m,n}(s,u)|^2\,du$$

$$+\frac{K^2}{\beta_2}E\int_s^T |Z_n(s,u)-Z_m(s,u)|^2\,du.$$

Consequently, by (3.26) we have

$$J_4\leqslant\eta^2\frac{C}{N^{2(1-\alpha)}} \tag{3.27}$$

$$+[3(\beta_2+3)\eta^2+1]\,E\int_s^T \left|(f_n-f)(s,u,Y_n(u),Z_n(s,u))\right|^2\mathbf{1}_{\overline{A}^N_{m,n}}(s,u)\,du$$

$$+[3(\beta_2+3)\eta^2+1]\,E\int_s^T \left|(f_m-f)(s,u,Y_m(u),Z_m(s,u))\right|^2\mathbf{1}_{\overline{A}^N_{m,n}}(s,u)\,du$$

$$+6(\beta_2+3)E\int_s^T (T-u)\,du\int_u^T \left|(f_n-f)(u,v,Y_n(v),Z_n(u,v))\right|^2\mathbf{1}_{\overline{A}^N_{m,n}}(u,v)\,dv$$

$$+6(\beta_2+3)E\int_s^T (T-u)\,du\int_u^T \left|(f_m-f)(u,v,Y_m(v),Z_m(u,v))\right|^2\mathbf{1}_{\overline{A}^N_{m,n}}(u,v)\,dv$$

$$+[12(\beta_2+3)\eta^2+1]\,L_N^2 E\int_s^T |Y_n(u)-Y_m(u)|^2\,du$$

$$+[6(\beta_2+3)\eta^2+1/\beta_2]\,K^2 E\int_s^T |Z_n(s,u)-Z_m(s,u)|^2\,du$$

$$+12(\beta_2+3)K^2\,TE\int_s^T du\int_u^T |Z_n(u,v)-Z_m(u,v)|^2\,dv.$$

We have

$$(3.28)\qquad J_5 \leqslant \beta_3 E\int_s^T |Y_n(u)-Y_m(u)|^2\,du + \frac{K^2}{\beta_3} E\int_s^T |Z_n(s,u)-Z_m(s,u)|^2\,du.$$

Consequently, from (3.21)–(3.28) we deduce that

$$\begin{aligned}
&E|Y_n(s)-Y_m(s)|^2 + E\int_s^T |Z_n(s,u)-Z_m(s,u)|^2\,du\\
&\leqslant \left(3+\beta_1+\beta_3+2L_N+[1+12(\beta_2+4)\eta^2]\right)L_N^2 E\int_s^T |Y_n(u)-Y_m(u)|^2\,du\\
&+\left(\frac{1}{\beta_1}+\frac{1}{\beta_2}+\frac{1}{\beta_3}+6(\beta_2+4)\eta^2\right)K^2 E\int_s^T |Z_n(s,u)-Z_m(s,u)|^2\,du\\
&+\left(2+3(\beta_2+4)\eta^2\right)E\int_s^T \left|(f_n-f)\left(s,u,Y_n(u),Z_n(s,u)\right)\right|^2 \mathbf{1}_{\bar{A}^N_{m,n}}(s,u)\,du\\
&+\left(2+3(\beta_2+4)\eta^2\right)E\int_s^T \left|(f_m-f)\left(s,u,Y_m(u),Z_m(s,u)\right)\right|^2 \mathbf{1}_{\bar{A}^N_{m,n}}(s,u)\,du\\
&+6(\beta_2+4)(T-s)\left[\varrho_N^2(f_n-f)+\varrho_N^2(f_m-f)\right]\\
&+12(\beta_2+4)K^2\,TE\int_s^T du\int_u^T |Z_n(u,v)-Z_m(u,v)|^2\,dv\\
&+\frac{C}{N^{2(1-\alpha)}}(1+\eta^2).
\end{aligned}$$

Let us choose $\beta_1=\beta_2=\beta_3=8K^2$ and define

$$U_{m,n}(t)=E\int_t^T |Y_n(s)-Y_m(s)|^2\,ds,\qquad V_{n,m}(t)=E\int_t^T |Z_n(t,s)-Z_m(t,s)|^2\,ds.$$

Then we have

$$\begin{aligned}
(3.29)\qquad &-\frac{d}{ds}\left(\exp(K_1 s)\,U_{m,n}(s)\right)+\frac{1}{2}\exp(K_1 s)\,V_{m,n}(s)\\
&\leqslant K_2 E\int_s^T \left|(f_n-f)\left(s,u,Y_n(u),Z_n(s,u)\right)\right|^2 \exp(K_1 s)\,\mathbf{1}_{\bar{A}^N_{m,n}}(s,u)\,du\\
&+K_2 E\int_s^T \left|(f_m-f)\left(s,u,Y_m(u),Z_m(s,u)\right)\right|^2 \exp(K_1 s)\,\mathbf{1}_{\bar{A}^N_{m,n}}(s,u)\,du\\
&+K_3\left[\varrho_N^2(f_n-f)+\varrho_N^2(f_m-f)\right]\exp(K_1 s)+K_4\int_s^T \exp(K_1 u)\,V_{m,n}(u)\,du\\
&+\frac{C}{N^{2(1-\alpha)}}\exp(K_1 s),
\end{aligned}$$

where $K_1 = 3+16K^2+2L_N+2L_N^2$, K_2, K_3, K_4 and C are constants depending only on K, T, and ξ. Integrating (3.29) from t to T, we obtain

$$\text{(3.30)}\qquad \exp(K_1 t)\, U_{m,n}(t)+\frac{1}{2}\int_t^T \exp(K_1 s)\, V_{m,n}(s)\, ds$$

$$\leqslant (K_2+K_3 T)[\varrho_N^2(f_n-f)+\varrho_N^2(f_m-f)]\exp(K_1 T)+K_4\int_t^T ds\int_s^T \exp(K_1 s)\, V_{m,n}(r)\, dr$$

$$+\frac{C}{K_1 N^{2(1-\alpha)}}\exp(K_1 T).$$

From the Gronwall inequality we deduce that

$$\text{(3.31)}\qquad \int_t^T \exp(K_1 s)\, V_{m,n}(s)\, ds$$

$$\leqslant \left((K_2+K_3 T)[\varrho_N^2(f_n-f)+\varrho_N^2(f_m-f)]\exp(K_1 T)+\frac{C}{K_1 N^{2(1-\alpha)}}\exp(K_1 T)\right)$$

$$\times \exp(K_4 T).$$

In view of the condition (A), passing to the limit successively for N, n and m in (3.30) and (3.31), we have

$$\int_t^T \exp(K_1 s)\, V_{m,n}(s)\, ds \to 0 \qquad \text{and} \qquad U_{m,n}(t)\to 0.$$

Therefore, $(Y_n, Z_n)_{n\geqslant 1}$ is a Cauchy sequence in the Banach space $M^2([t, T]; \mathbf{R}^k) \times M^2([T-\eta, T]\times[t, T]; \mathbf{R}^{k\times d})$.

We put

$$Y(s) = \lim_n Y_n(s) \qquad \text{and} \qquad Z(t, s) = \lim_n Z_n(t, s).$$

On the other hand, if we put

$$A_n^N = \{(\omega, t, s)\in\Omega\times\mathcal{D}_\eta,\ 1+|Y(u)|+|Z(s, u)|+|Y_n(u)|+|Z_n(s, u)| > N\},$$

$$\bar{A}_n^N = (\Omega\times\mathcal{D}_\eta)\backslash A_n^N,$$

then

$$\int_t^T ds\, E\int_s^T \big|f_n(s, u, Y_n(u), Z_n(s, u))-f(s, u, Y(u), Z(s, u))\big|^2 du$$

$$\leqslant \int_t^T ds\, E\int_s^T \big|f_n(s, u, Y_n(u), Z_n(s, u))-f(s, u, Y(u), Z(s, u))\big|^2 \mathbf{1}_{A_n^N}(s, u)\, du$$

$$+2\int_t^T ds\, E\int_s^T \big|(f_n-f)(s, u, Y_n(u), Z_n(s, u))\big|^2 \mathbf{1}_{\bar{A}_n^N}(s, u)\, du$$

$$+2\int_t^T ds E\int_s^T \left|f(s, u, Y_n(u), Z_n(s, u))-f(s, u, Y(u), Z(s, u))\right|^2 \mathbf{1}_{\overline{A_n^N}}(s, u)\, du$$

$$\leqslant \frac{C}{N^{2(1-\alpha)}}+2\varrho_N^2(f_n-f)+4L_N^2 E\int_t^T |Y_n(s)-Y(s)|^2 ds$$

$$+4K^2 E\int_s^T ds\int_u^T |Z_n(s, u)-Z(s, u)|^2 du.$$

Passing to the limit successively for N and n, we obtain

$$\int_t^T dsE\int_s^T \left|f_n(s, u, Y_n(u), Z_n(s, u))\right.$$
$$\left.-f(s, u, Y(u), Z(s, u))\right|^2 du \to 0 \qquad \text{for all } t\in[T-\eta, T].$$

Then, taking the limit in (3.1), we see that (Y, Z) solves equation (1.1) for $(t, s)\in [T-\eta, T]\times[t, T]$.

From the above calculations we know that for $(t, s)\in[T-\eta, T]\times[t, T]$ there exists unique $Y(T-\eta)$. Now, for $(t, s)\in[T-2\eta, T-\eta]\times[t, T-\eta]$, we consider the equation

$$Y_n(t)+\int_t^{T-\eta} f_n(t, s, Y_n(s), Z_n(t, s))\,ds$$
$$+\int_t^{T-\eta}\left[g(t, s, Y_n(s))+Z_n(t, s)\right]dW(s) = Y(T-\eta).$$

With the same argument as above, one can prove that $(Y_n, Z_n)_{n\in N}$ is a Cauchy sequence in the Banach space $M^2([T-2\eta, T-\eta]; \mathbf{R}^k)\times M^2([T-2\eta, T-\eta]\times [t, T-\eta]; \mathbf{R}^{k\times d})$. One can prove that its limit is the unique solution of the Volterra equation with data (ξ, f, g) for $(t, s)\in[T-2\eta, T-\eta]\times[t, T-\eta]$. Thus, we can prove the existence by continuing this procedure. ∎

4. STABILITY RESULTS FOR BSNVIE WITH LOCAL LIPSCHITZ DRIFT

In this section, we prove a stability result for backard stochastic nonlinear Volterra integral equations assuming local Lipschitz drift. Let $(\xi_n)_{n\in N^*}$ be a sequence of random variables and $(f_n, g_n)_{n\geqslant 1}$ a sequence of processes which fulfill assumptions of Theorem 3.3. We denote by (Y_n, Z_n) the unique solution of the BSDE of Volterra type with data (ξ_n, f_n, g_n). Moreover, we consider the following assumption:

(A4) For each $N\in N^*\backslash\{1\}$,
(i) $\varrho_N(f_n-f_0)\to 0$ as $n\to+\infty$,
(ii) $\pi(g_n-g_0)\to 0$ as $n\to+\infty$,

(iii) $E|\xi_n-\xi_0|^2 \to 0$ as $n\to+\infty$,

where

$$\pi(g_n-g_0)=E\Big(\int_{\mathscr{D}}\sup_{y\in R^k}|g_n(t,s,y)-g_0(t,s,y)|^2\,ds\,dt\Big)^{1/2}$$

THEOREM 4.1. *Assume* (A1)–(A4) *and* (A) *hold true. Then*

$$(Y_n,Z_n)\to(Y_0,Z_0)\ \textit{in}\ M^2(t,T,R^k)\times M^2(\mathscr{D},R^{k\times d})\qquad \textit{as}\ n\to+\infty.$$

Proof. Let $\eta>0$ (to be precised later). For each $(t,s)\in[T-\eta,T]\times[t,T]$ it follows from Lemma 2.1 of [21] that

$$E|Y_n(s)-Y_0(s)|^2+E\int_s^T|Z_n(s,u)-Z_0(s,u)|^2\,du=E|\xi_n-\xi_0|^2$$

$$+2E\int_s^T\big\langle f_n(s,u,Y_n(u),Z_n(s,u))-f_0(s,u,Y_0(u),Z_0(s,u)),\,Y_n(u)-Y_0(u)\big\rangle\,du$$

$$-2E\int_s^T\big\langle f_n(s,u,Y_n(u),Z_n(s,u))-f_0(s,u,Y_0(u),Z_0(s,u)),\,I_{n,0}(u)\big\rangle\,du$$

$$-2E\int_s^T\big\langle g_n(s,u,Y_n(u))-g_0(s,u,Y_0(u)),\,Z_n(s,u)-Z_0(s,u)\big\rangle\,du$$

$$-E\int_s^T\big|g_n(s,u,Y_n(u))-g_0(s,u,Y_0(u))\big|^2\,du.$$

Therefore we obtain

$$E|Y_n(s)-Y_0(s)|^2+E\int_s^T|Z_n(s,u)-Z_0(s,u)|^2\,du$$

$$\leqslant E|\xi_n-\xi_0|^2$$

$$+2E\int_s^T\big|\big\langle f_n(s,u,Y_n(u),Z_n(s,u))-f_0(s,u,Y_0(u),Z_0(s,u)),\,Y_0(u)-Y_0(u)\big\rangle\big|\,du$$

$$+2E\int_s^T\big|\big\langle f_n(s,u,Y_n(u),Z_n(s,u))-f_0(s,u,Y_0(u),Z_0(s,u)),\,I_{n,0}(s,u)\big\rangle\big|\,du$$

$$+2E\int_s^T\big|\big\langle g_n(s,u,Y_n(u))-g_0(s,u,Y_0(u)),\,Z_n(s,u)-Z_0(s,u)\big\rangle\big|\,du,$$

where

$$I_{n,0}(s,u)=\int_u^T\big(f_n(u,v,Y_n(v),Z_n(u,v))-f_0(u,v,Y_0(v),Z_0(u,v))\big)\,dv$$

$$-\int_u^T\big(f_n(s,v,Y_n(v),Z_n(s,v))-f_0(s,v,Y_0(v),Z_0(s,v))\big)\,du.$$

For each $N > 1$, let us consider L_N, the Lipschitz constant of f in the ball $B(0, N)$ of $\mathbf{R}^k$, and put

$$D_{n,0}^N = \{(\omega, s, u) \in \Omega \times \mathscr{D}_\eta,\ |Y_n(s)| + |Z_n(s, u)| + |Y_0(u)| + |Z_0(s, u)| \geqslant N\},$$
$$\bar{D}_{n,0}^N = (\Omega \times \mathscr{D}_\eta) \backslash D_{n,0}^N.$$

The same procedure as in the proof of the existence part in Theorem 3.3 yields

$$|Y_n(s) - Y_0(s)|^2 + \mathbf{E}\int_s^T |Z_n(s, u) - Z_0(s, u)|^2\, du$$

$$\leqslant \mathbf{E}|\xi_n - \xi_0|^2 + \big(2 + \beta_1 + \beta_3 + 2L_N + (1 + 8(\beta_2 + 3))\eta^2\big) L_N^2\, \mathbf{E}\int_s^T |Y_n(u) - Y_0(u)|^2\, du$$

$$+ \left[\left(\frac{1}{\beta_1} + \frac{1}{\beta_2} + \frac{1}{\beta_3} + 4(\beta_2 + 3)\eta^2\right) K^2 + \frac{1}{\beta_4}\right] \mathbf{E}\int_s^T |Z_n(s, u) - Z_0(s, u)|^2\, du$$

$$+ \big(3 + 2(\beta_2 + 3)\eta^2\big)\mathbf{E}\int_s^T \big|(f_n - f_0)(s, u, Y_n(u), Z_n(s, u))\big|^2 \mathbf{1}_{\bar{D}_{n,0}^N}(s, u)\, du$$

$$+ 4(\beta_2 + 3)\mathbf{E}\int_s^T (T - u)\, du \int_u^T \big|(f_n - f_0)(u, v, Y_n(v), Z_n(u, v))\big|^2 \mathbf{1}_{\bar{D}_{n,0}^N}(u, v)\, dv$$

$$+ \beta_4 \mathbf{E}\int_s^T \big|(g_n - g_0)(s, u, Y_n(u))\big|^2\, du$$

$$+ 8(\beta_2 + 3) K^2\, T\mathbf{E}\int_s^T du \int_u^T |Z_n(u, v) - Z_0(u, v)|^2\, dv + (\eta^2 + 1)\frac{C}{N^{2(1-\alpha)}}.$$

Let us choose $\beta_1 = \beta_2 = \beta_3 = 12K^2$, $\beta_4 = 8$, and $\eta < 1/24K^2$. If we define

$$U_{n,0}(t) = \mathbf{E}\int_t^T |Y_n(s) - Y_0(s)|^2\, ds \quad \text{and} \quad V_{n,0}(t) = \mathbf{E}\int_t^T |Z_n(t, s) - Z_0(t, s)|^2\, ds,$$

then we obtain

$$-\frac{d}{ds}\big(\exp(K_1 s)\, U_{n,0}(s)\big) + \frac{1}{2}\exp(K_1 s)\, V_{n,0}(s)$$
$$\leqslant \mathbf{E}\big(\exp(K_1 s)|\xi_n - \xi_0|^2\big)$$
$$+ 4\mathbf{E}\exp(K_1 s)\int_s^T \big|(f_n - f_0)(s, u, Y_n(u), Z_n(s, u))\big|^2 \mathbf{1}_{\bar{D}_{n,0}^N}(s, u)\, du$$
$$+ 4(12K^2 + 3)\mathbf{E}\exp(K_1 s)$$
$$\times \int_s^T (T - u)\, du \int_u^T \big|(f_n - f_0)(u, v, Y_n(v), Z_n(u, v))\big|^2 \mathbf{1}_{\bar{D}_{n,0}^N}(u, v)\, dv$$

$$+8E\exp(K_1 s)\int_s^T |(g_n-g_0)(s,u,Y_n(u))|^2\,du$$

$$+8(12K^2+3)K^2\,TE\exp(K_1 s)\int_s^T V_{n,0}(u)\,du+\exp(K_1 s)\frac{C}{N^{2(1-\alpha)}},$$

where $K_1 = 2+24K^2+2L_N+2L_N^2$, K_2, K_3 and C are constants depending only on K, T, and ξ_0. The rest of the proof is identical to that of the existence part of Theorem 3.3. ∎

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